Math 1433

31 October 2023

What is $\dfrac{1}{2}+\dfrac{2}{3}$?

Answer: $\dfrac{7}{6}$

(not $\dfrac{1+2}{2+3}$)

Easy calculations:

- $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
- $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$
- $s \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} sa \\ sb \end{bmatrix}$

We will never use these:

- $\frac{a+c}{b+d}$
- $\begin{bmatrix} ac \\ bd \end{bmatrix}$

Surprising (?) formulas:

- $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
- $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd$
- $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$
- $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} bf-ce \\ cd-af \\ ae-bd \end{bmatrix}$

Perpendicular

Previously

Two vectors are called **orthogonal** if their dot product is zero.

- For non-zero vectors, this means they are **perpendicular** (or **normal**). Why?

$$\cos(\text{angle}) = 0 \quad \leftrightarrow \quad \text{angle} = 90°$$

- The zero vector is orthogonal to every vector.

Give an example of a vector that is perpendicular to v = i + 9j + 4k.

To find a vector that is perpendicular to the vector $\mathbf{v} = \mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$, you can use the fact that two vectors are perpendicular if their dot product is equal to zero.

So, you need to find a vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ such that:

true

$$\mathbf{u} \cdot \mathbf{v} = 0$$

The dot product of two vectors is calculated as follows:

$$\mathbf{u} \cdot \mathbf{v} = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}) = a \cdot 1 + b \cdot 9 + c \cdot 4$$

For the dot product to be zero, you can choose any values of a, b, and c that satisfy the equation:

$$a + 9b + 4c = 0$$

Here's an example of a vector that is perpendicular to $\mathbf{v}$:

If you let $a = -36$, $b = 4$, and $c = 9$, you get:

$$\mathbf{u} = -36\mathbf{i} + 4\mathbf{j} + 9\mathbf{k}$$

wrong!

Now, when you take the dot product:

$$\mathbf{u} \cdot \mathbf{v} = (-36\mathbf{i} + 4\mathbf{j} + 9\mathbf{k}) \cdot (\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}) = -36 \cdot 1 + 4 \cdot 9 + 9 \cdot 4 = 0$$

So, the vector $-36\mathbf{i} + 4\mathbf{j} + 9\mathbf{k}$ is perpendicular to $\mathbf{v} = \mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$. You could also find other vectors that are perpendicular to $\mathbf{v}$ by choosing different values for a, b, and c as long as they satisfy the equation $a + 9b + 4c = 0$.

So, the vector $-36\mathbf{i} + 4\mathbf{j} + 9\mathbf{k}$ is perpendicular to $\mathbf{v} = \mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$. You could also find other vectors that are perpendicular to $\mathbf{v}$ by choosing different values for a, b, and c as long as they satisfy the equation $a + 9b + 4c = 0$.

Calculate (-36i + 4j + 9k) · (i + 9k + 4k) again.

Apologies for the typographical error in my previous response. Let's correct the calculation for the dot product of $(-36\mathbf{i} + 4\mathbf{j} + 9\mathbf{k})$ and $(\mathbf{i} + 9\mathbf{j} + 4\mathbf{k})$:

$$(-36\mathbf{i} + 4\mathbf{j} + 9\mathbf{k}) \cdot (\mathbf{i} + 9\mathbf{j} + 4\mathbf{k})$$

Now, let's compute the dot product correctly:

$$(-36\mathbf{i} + 4\mathbf{j} + 9\mathbf{k}) \cdot (\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}) = (-36 \cdot 1) + (4 \cdot 9) + (9 \cdot 4) = -36 + 36 + 36 = 0$$

So, the corrected dot product is indeed 0, confirming that $(-36\mathbf{i} + 4\mathbf{j} + 9\mathbf{k})$ is perpendicular to $(\mathbf{i} + 9\mathbf{j} + 4\mathbf{k})$.

What is -36 + 36 + 36 ?

I apologize for the mistake. The correct calculation of $-36 + 36 + 36$ is:

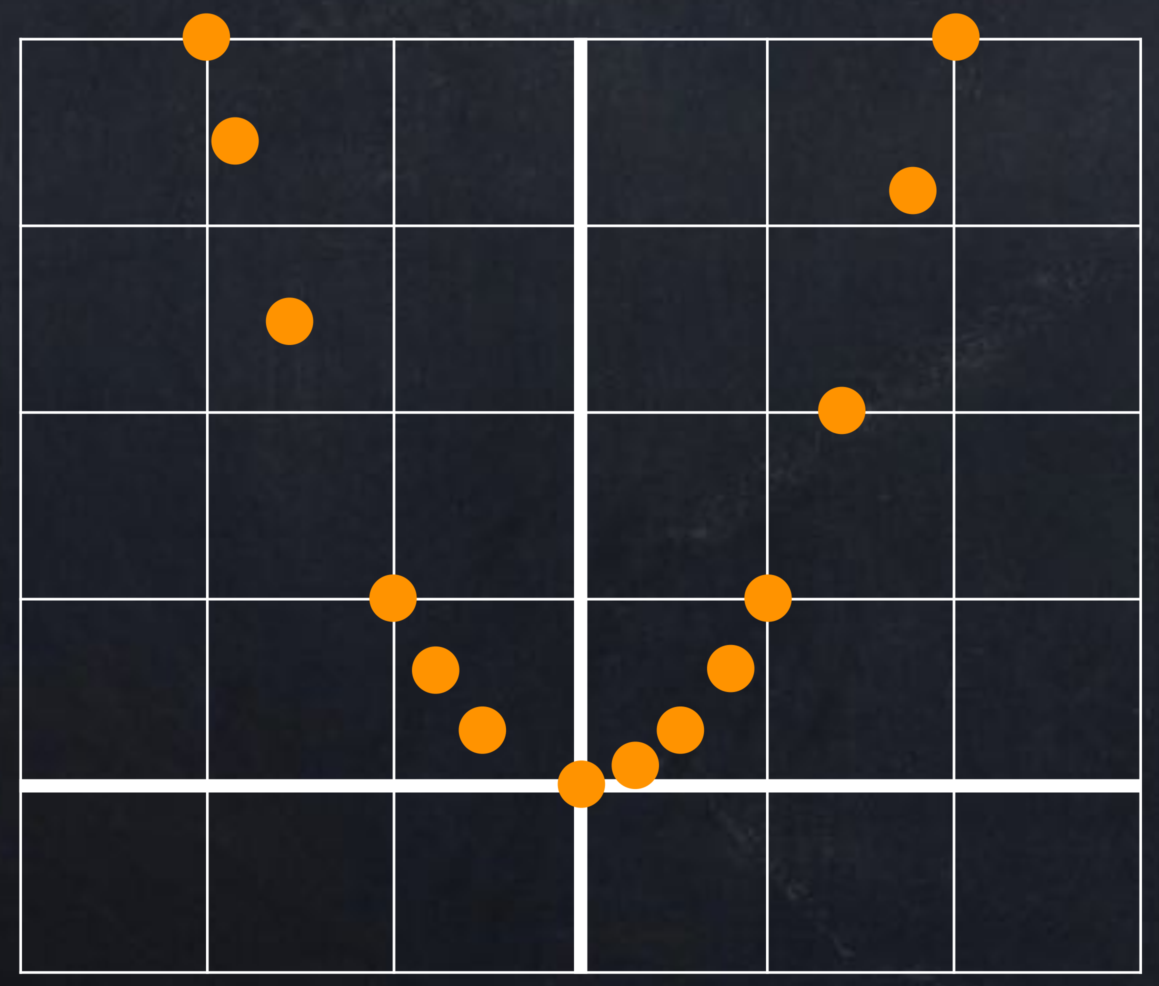
$$-36 + 36 + 36 = 0$$

So, the dot product of $(-36\mathbf{i} + 4\mathbf{j} + 9\mathbf{k})$ and $(\mathbf{i} + 9\mathbf{j} + 4\mathbf{k})$ is indeed 0, as previously stated.

Equations of shapes

There are two ways to think about, for example, $y = x^2$.

- For each specific x-value, you get a y-value, and this tells you one point on the curve.
- For any point (x, y) anywhere, we test whether $y = x^2$ is true for those numbers.



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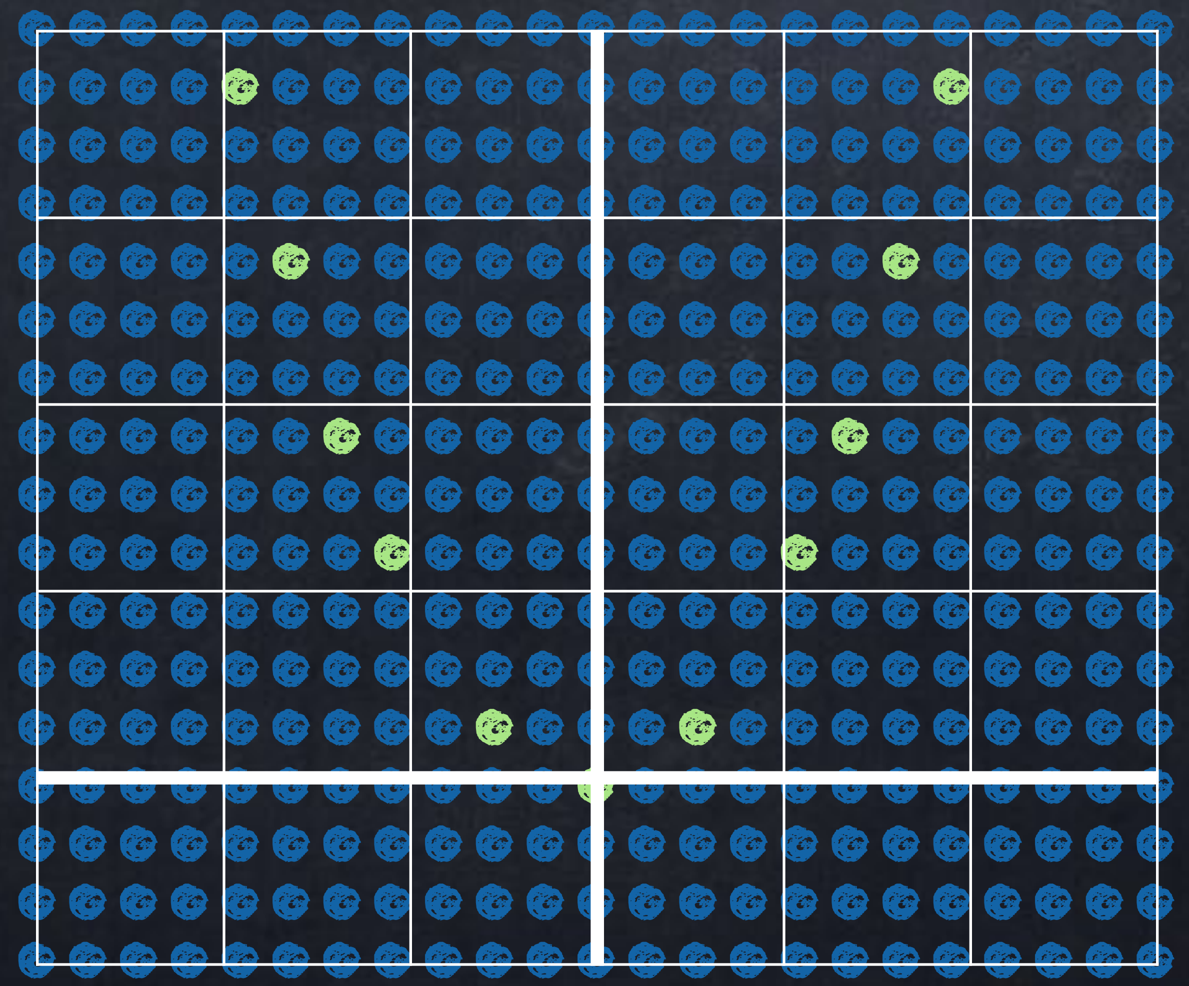

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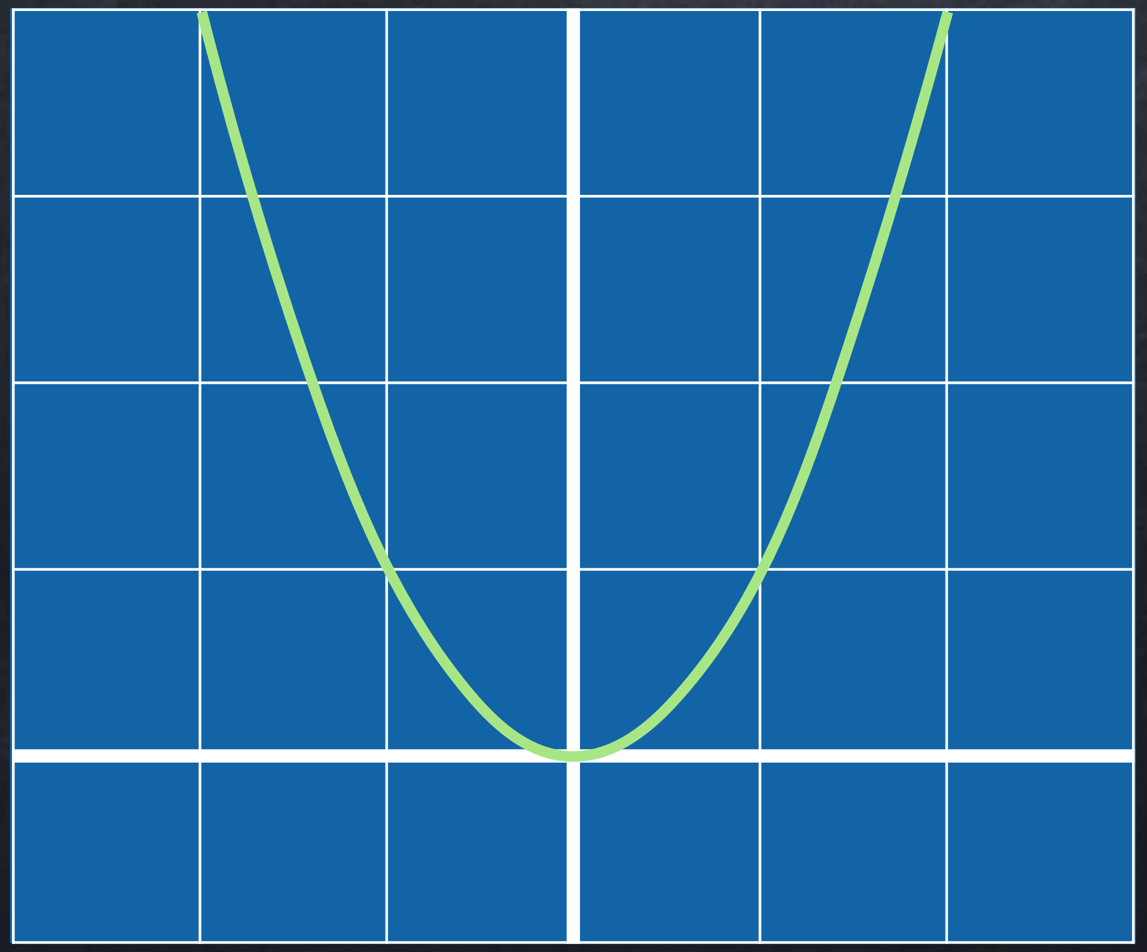
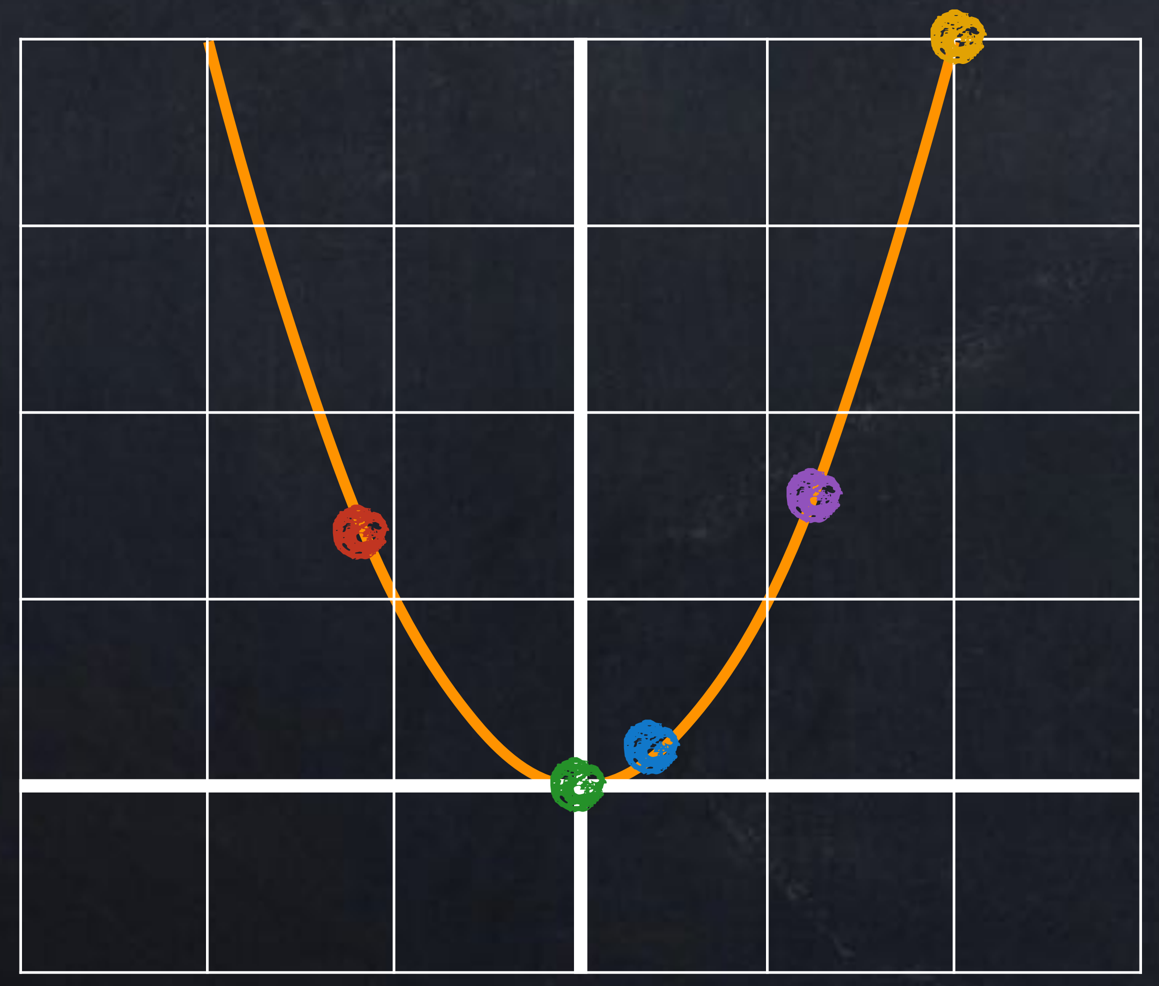


Equations of shapes

Instead of $y = x^2$, we can also describe the same shape using “parametric equations” in many ways. One example is

$$x = \frac{1}{5}t^3, \qquad y = \frac{1}{25}t^6$$

where t is a parameter that can take any value in $\mathbb{R}$. (For this parabola, parametric equations are unnecessary, but for some shapes it is very helpful.)



$t = 0 \to (x,y) = (0, 0)$

$t = 1 \to (x,y) = (0.2, 0.04)$

$t = 2 \to (x,y) = (1.6, 2.56)$

$t = -1.9 \to (-1.1664, 1.3605)$

$t = 2.1544 \to (x,y) = (2,4)$

Previously

The line through point (x_0, y_0, z_0) parallel to vector $\overrightarrow{D} = [a, b, c]$ can be described by the single vector equation

$$\vec{r} = \vec{p} + t\,\vec{d}$$

where $\vec{p} = [x_0, y_0, z_0]$, or by several scalar equations:

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

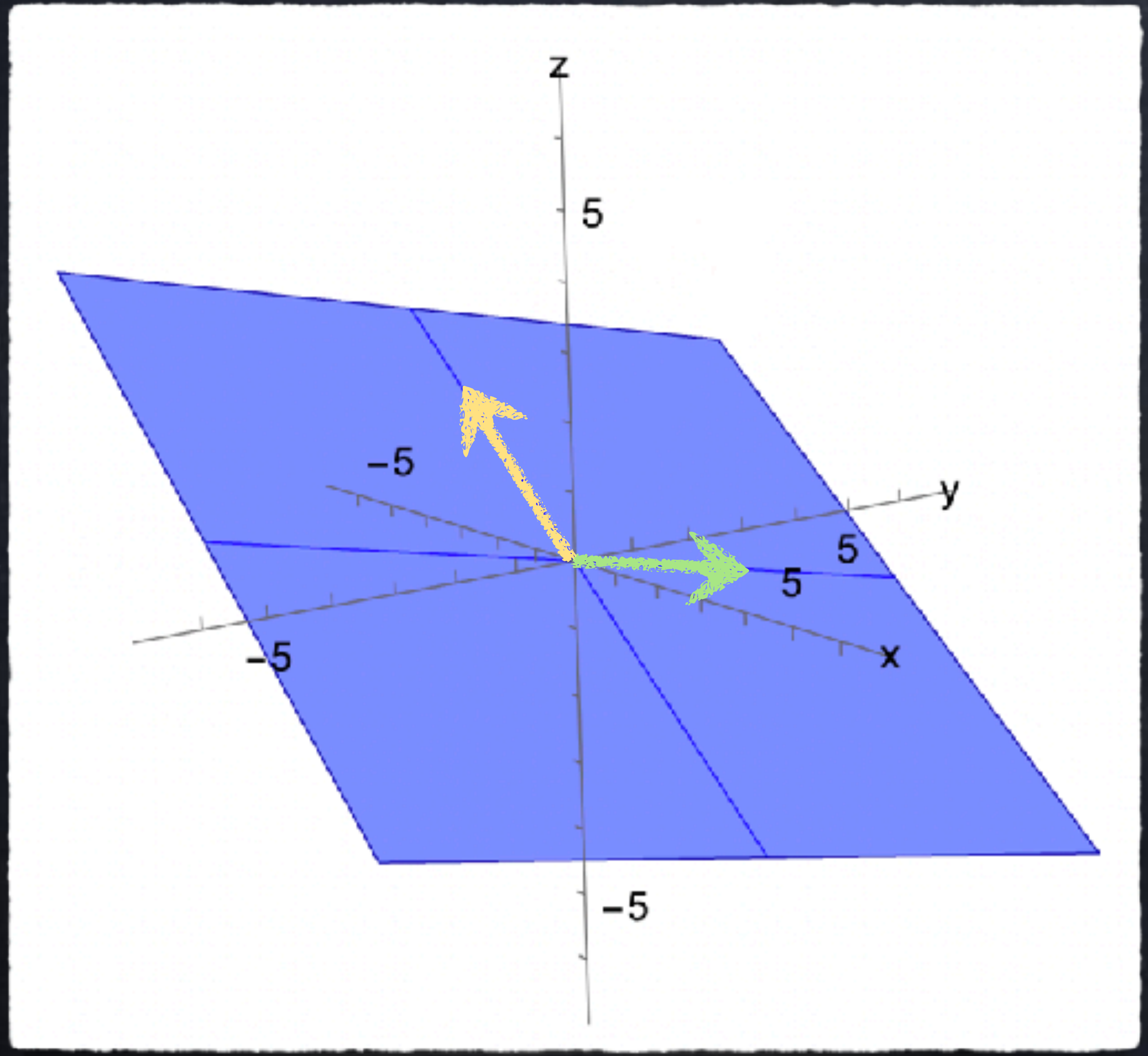
In both cases, the variable t is a parameter (sometimes s is used instead).

The vector $\vec{d}$ is called a **direction vector** for the line.

Planes

We usually use a parameter (t) to describe a line in 3D space.

A plane in space can be described by parametric equations, but we require two parameters!



The plane through the origin parallel to both $\vec{a}$ and $\vec{b}$ is described by

$$[x, y, z] = t\vec{a} + s\vec{b}.$$

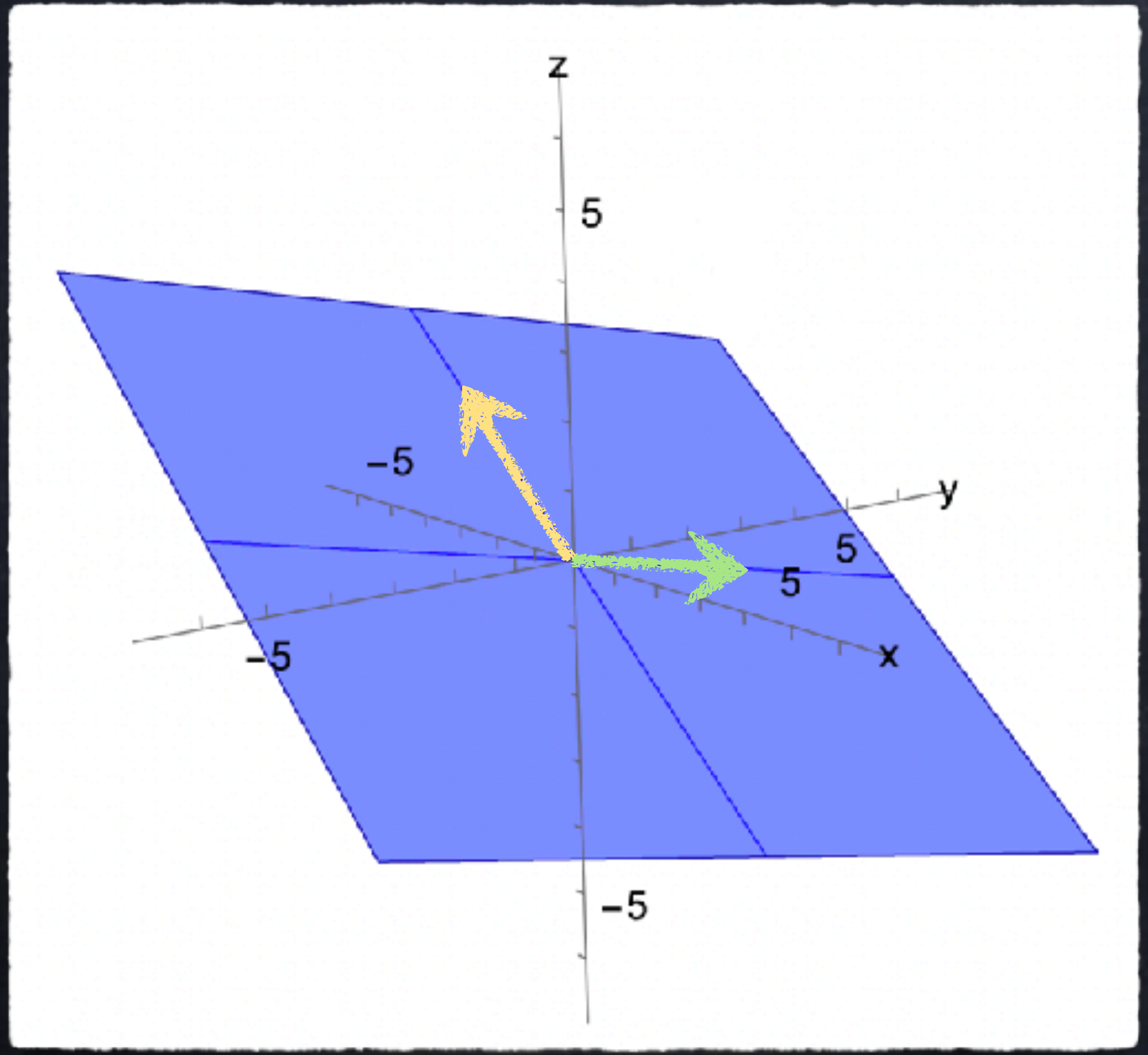
If the plane goes through *point* $\vec{p}$ (not necessarily the origin), then it's

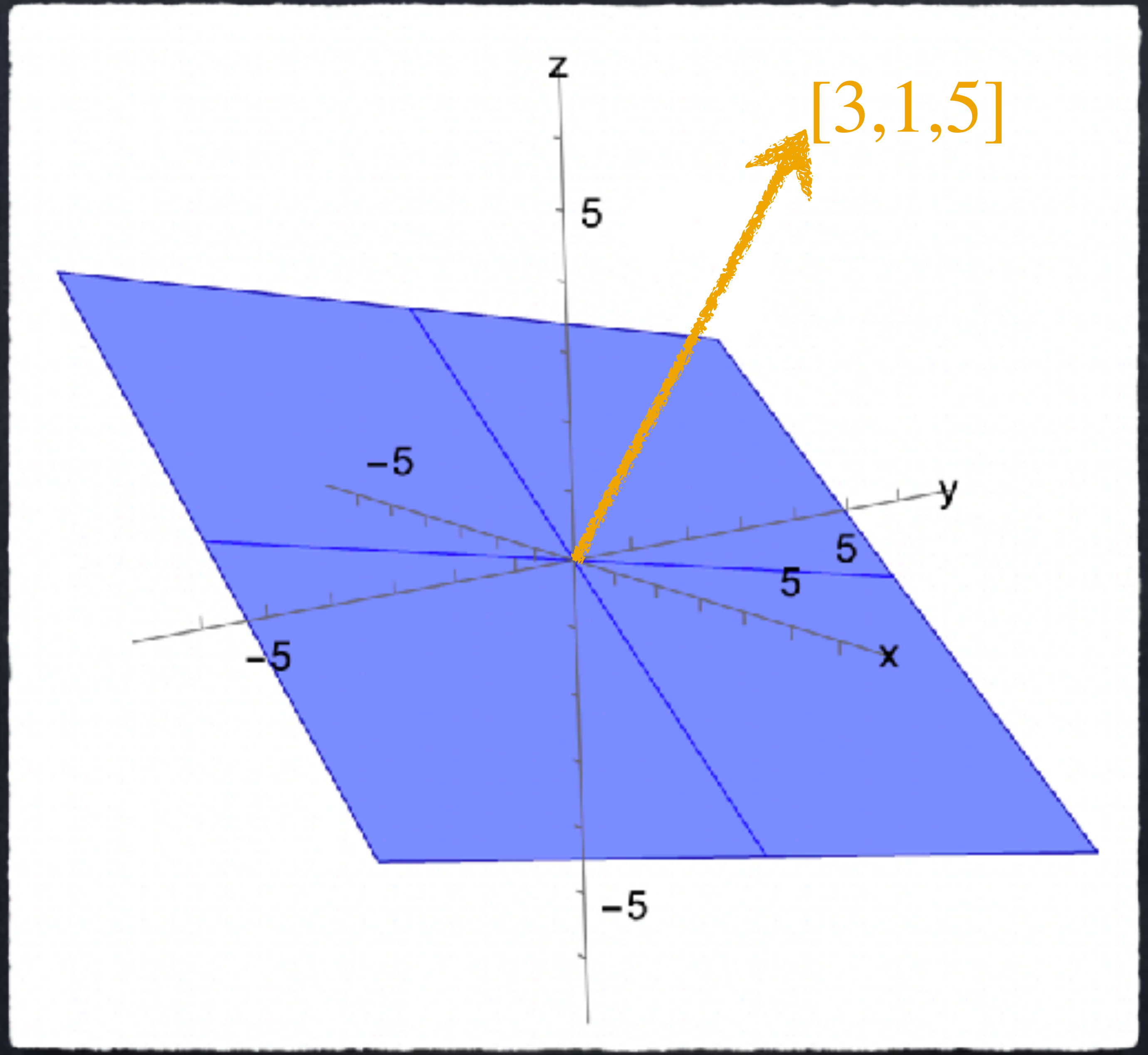
$$[x, y, z] = \vec{p} + t\vec{a} + s\vec{b}.$$

Planes

We can also describe the plane below as being perpendicular to a single vector.

How can we get an equation from this?





A vector perpendicular to a plane is called a **normal vector** for the plane.

Is $(-1, 2, 8)$ is on the plane through the origin with normal vector $[3, 1, 5]$?

IF the point $(-1, 2, 8)$ is on the plane through $(0,0,0)$ with normal vector $[3, 1, 5]$ then

- the arrow from $(0,0,0)$ to $(-1, 2, 8)$ is perpendicular to $[3, 1, 5]$.
- the vector $[-1, 2, 8]$ is perpendicular to $[3, 1, 5]$.
- the dot product $[3, 1, 5] \cdot [-1, 2, 8]$ equals 0.
- $3(-1) + 1(2) + 5(8) = 0$.
- $39 = 0$.

So we know that $(-1, 2, 8)$ is *not* on this plane.

IF the point $(4, -7, -1)$ is on the plane through $(0,0,0)$ with normal vector $[3, 1, 5]$ then

- the arrow from $(0,0,0)$ to $(4, -7, -1)$ is perpendicular to $[3, 1, 5]$.
- the vector $[4, -7, -1]$ is perpendicular to $[3, 1, 5]$.
- the dot product $[3, 1, 5] \cdot [4, -7, -1]$ equals 0.
- $3(4) + 1(-7) + 5(-1) = 0$.
- $0 = 0$.

So we know that $(4, -7, -1)$ *is* on this plane.

If the point (x, y, z) is on the plane through $(0,0,0)$ with normal vector $[3, 1, 5]$ then

- the arrow from $(0,0,0)$ to (x, y, z) is perpendicular to $[3, 1, 5]$.
- the vector $[x, y, z]$ is perpendicular to $[3, 1, 5]$.
- the dot product $[3, 1, 5] \cdot [x, y, z]$ equals 0.
- $3x + y + 5z = 0.$

So " $3x + y + 5z = 0$ " is the equation for the plane through the origin normal to $[3, 1, 5]$!

Remember,

$\vec{r}$ means $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The plane in 3D through $(0, 0, 0)$ with normal vector $\vec{n}$ has equation

$$\vec{n} \cdot \vec{r} = 0.$$

If $\vec{n}$ is [a,b,c], this eqn is ax+by+cz=0.

Task: Give an equation for the plane through $(0,0,0)$ and $(4,6,1)$ and $(8,7,2)$.

We first need a vector $\vec{n} = [a,b,c]$ that is perpendicular to $[4,6,1]$ and perpendicular to $[8,7,2]$.

OPTION 1: Find a solution to the system

$$4a + 6b + c = 0.$$
$$8a + 7b + 2c = 0.$$

OPTION 2: Use the cross-product:

$$\vec{n} = [4,6,1] \times [8,7,2] = [5,0,-20].$$

Answer: $[5,0,-20] \cdot [x,y,z] = 0$ means that

$$\boxed{5x - 20z = 0}$$

or $x - 4z = 0$ or similar equations. In fact $\vec{n}$ can be any non-zero scalar multiple of $[1,0,-4]$.

What if the plane does not go through $(0, 0, 0)$?

Is $(-9, 8, 2)$ is on the plane through $(4, 1, 1)$ with $\vec{n} = [3, 1, 5]$?

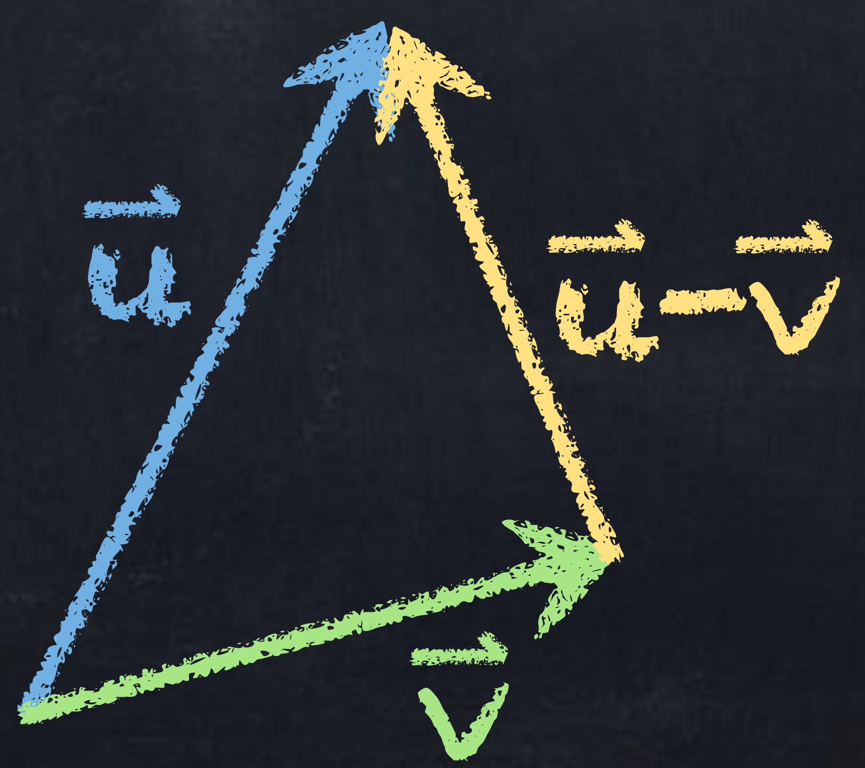
IF it is, then

- the arrow from $(4, 1, 1)$ to $(-9, 8, 2)$ is perpendicular to $[3, 1, 5]$.
  - *What is this arrow?*

This exact slide, including the giant text, was on 9.10.
A similar slide was shown on 16.10.

Pre-viously

Vector $\vec{u} - \vec{v}$ points from the end of $\vec{v}$ to the end of $\vec{u}$.



Note: The tails (start) of $\vec{u}$ and $\vec{v}$ must be at the same place to use this method.

What if the plane does not go through $(0, 0, 0)$?

Is $(-9, 8, 2)$ is on the plane through $(4, 1, 1)$ with $\vec{n} = [3, 1, 5]$?

IF it is, then

- the arrow from $(4, 1, 1)$ to $(-9, 8, 2)$ is perpendicular to $[3, 1, 5]$.
- the vector $\big([-9, 8, 2] - [4, 1, 1]\big)$ is perpendicular to $[3, 1, 5]$.
- the dot product $[3, 1, 5] \cdot \big([-9, 8, 2] - [4, 1, 1]\big)$ equals 0.
- $3(-9\ -\ 4) + 1(8\ -\ 1) + 5(2\ -\ 1) = 0.$

What if the plane does not go through $(0, 0, 0)$?

(x, y, z)

Is $(-9, 8, 2)$ is on the plane through $(4, 1, 1)$ with $\vec{n} = [3, 1, 5]$?

IF it is, then

- the arrow from $(4, 1, 1)$ to $(-9, 8, 2)$ is perpendicular to $[3, 1, 5]$.
- the vector $\big([-9, 8, 2] - [4, 1, 1]\big)$ is perpendicular to $[3, 1, 5]$.
- the dot product $[3, 1, 5] \cdot \big([-9, 8, 2] - [4, 1, 1]\big)$ equals 0.
- $3(-9 - 4) + 1(8 - 1) + 5(2 - 1) = 0.$

$3(x - 4) + 1(y - 1) + 5(z - 1) = 0$

What if the plane does not go through $(0, 0, 0)$?

Is (x, y, z) is on the plane through $(4, 1, 1)$ with $\vec{n} = [3, 1, 5]$?

IF it is, then

- $3(x - 4) + 1(y - 1) + 5(z - 1) = 0.$

The plane through (x_0, y_0, z_0) perpendicular to $\vec{n} = [a, b, c]$ can be described by the vector equation

$$\boxed{\vec{n} \cdot (\vec{r} - \vec{p}) = 0}$$

where $\vec{p} = [x_0, y_0, z_0]$. Using vector $\cdot$ and $-$, this can be re-written as

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

or, after a little algebra, as

$$\boxed{ax + by + cz = d}$$

where $d = \vec{n} \cdot \vec{p}$. The vector $\vec{n}$ is called a **normal vector** for the plane.

The “easy” line and plane tasks are

- from a point and normal vector, give an equation for a plane.
- from a point and direction vector, give and equation for a line.

The “hard” line and plane tasks is when you are not given a point and a normal/direction vector but must figure them out from other information.

There are also tasks asking about intersections. These usually involve solving a system of equations.

Give an equation *without vectors* for the plane through $(12, 4, -3)$ normal to $[-2, 8, 8]$.

With vectors,
$$\begin{bmatrix} -2 \\ 8 \\ 8 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 12 \\ 4 \\ -3 \end{bmatrix} \right) = 0$$

Without vectors,

$$-2(x - 12) + 8(y - 4) + 8(z + 3) = 0$$

or

$$-2x + 8y + 8z = -16$$

or

$$x - 4y - 4z = 8$$

Find the intersection of

- the line through $(4, 3, 10)$ with direction vector $[1, 2, -4]$

and

- the plane through $(5, 0, 6)$ with normal vector $[2, 6, 1]$.

Answer: (2, -1, 18)

Give an equation $\_\_x + \_\_y + \_\_z = \_\_$ for the plane through $(-2, 7, 6)$ parallel to both

Line 1: $x = 8 + t, \quad y = 9 + 4t, \quad z = -7 + 10t$

and

Line 2: $x = 10 + 3t, \quad y = 3t, \quad z = 7 + 7t.$

We need

- a point on the plane. Use (-2,7,6).
- a normal vector for the plane.
  - This vector will be perp. to both [1, 4, 10] and [3, 3, 7].
  - We can use the cross product
    [1, 4, 10] × [3, 3, 7] = [-2, 23, -9] for this.

From $-2(x+2) + 23(y-7) - 9(z-6) = 0$ we get $\boxed{-2x + 23y - 9z = 111}$

There are many more line and plane tasks on List 2.

Now for a new topic……

Transformations of vectors

The functions you study in school and in Analysis 1 are usually from $\mathbb{R}$ to $\mathbb{R}$, meaning the input and output are numbers.

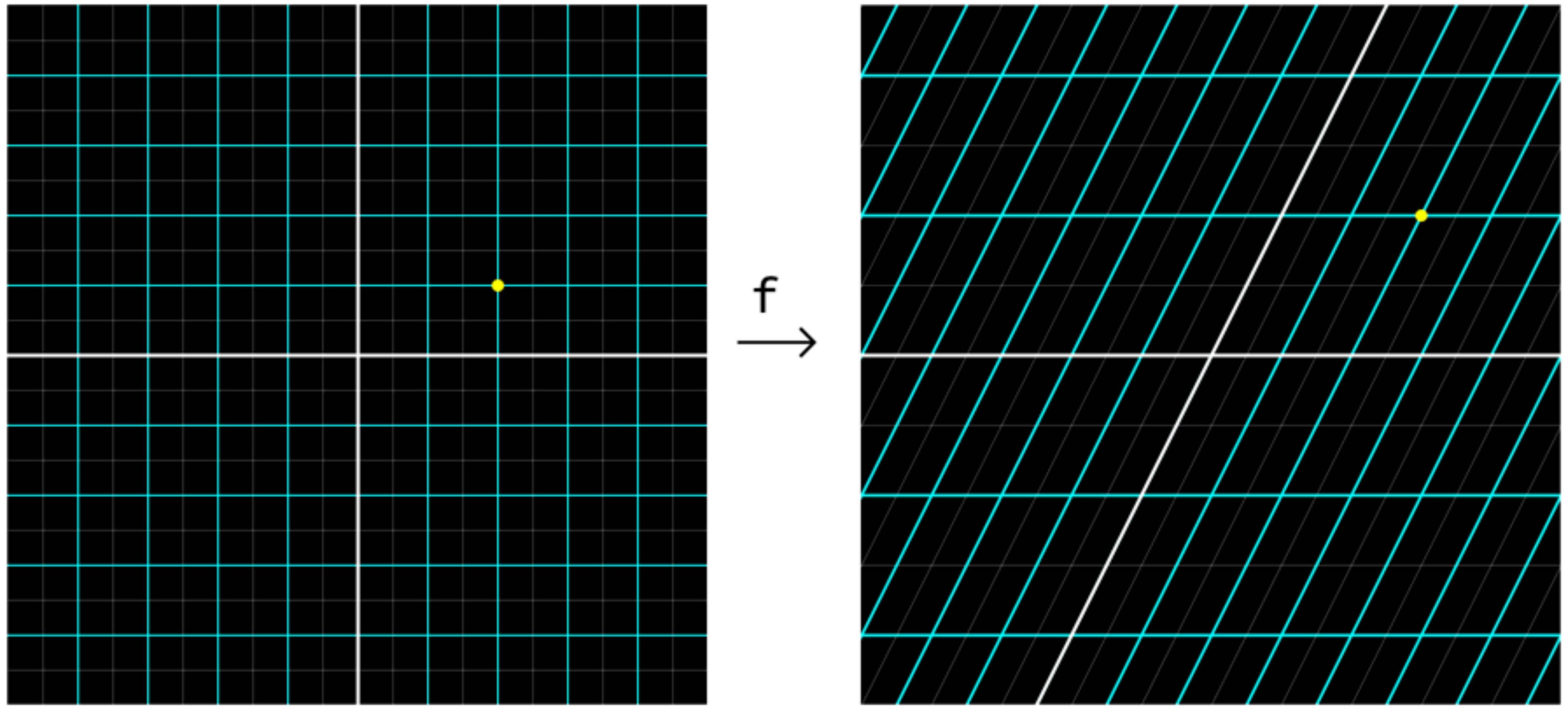
An example of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ could be $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ e^x \end{bmatrix}$.

We can also write that as

$$f(x\hat{\imath} + y\hat{\jmath}) = (x-y)\hat{\imath} + e^x\,\hat{\jmath} \qquad \text{or} \qquad f(x, y) = \left(x-y, e^x\right).$$

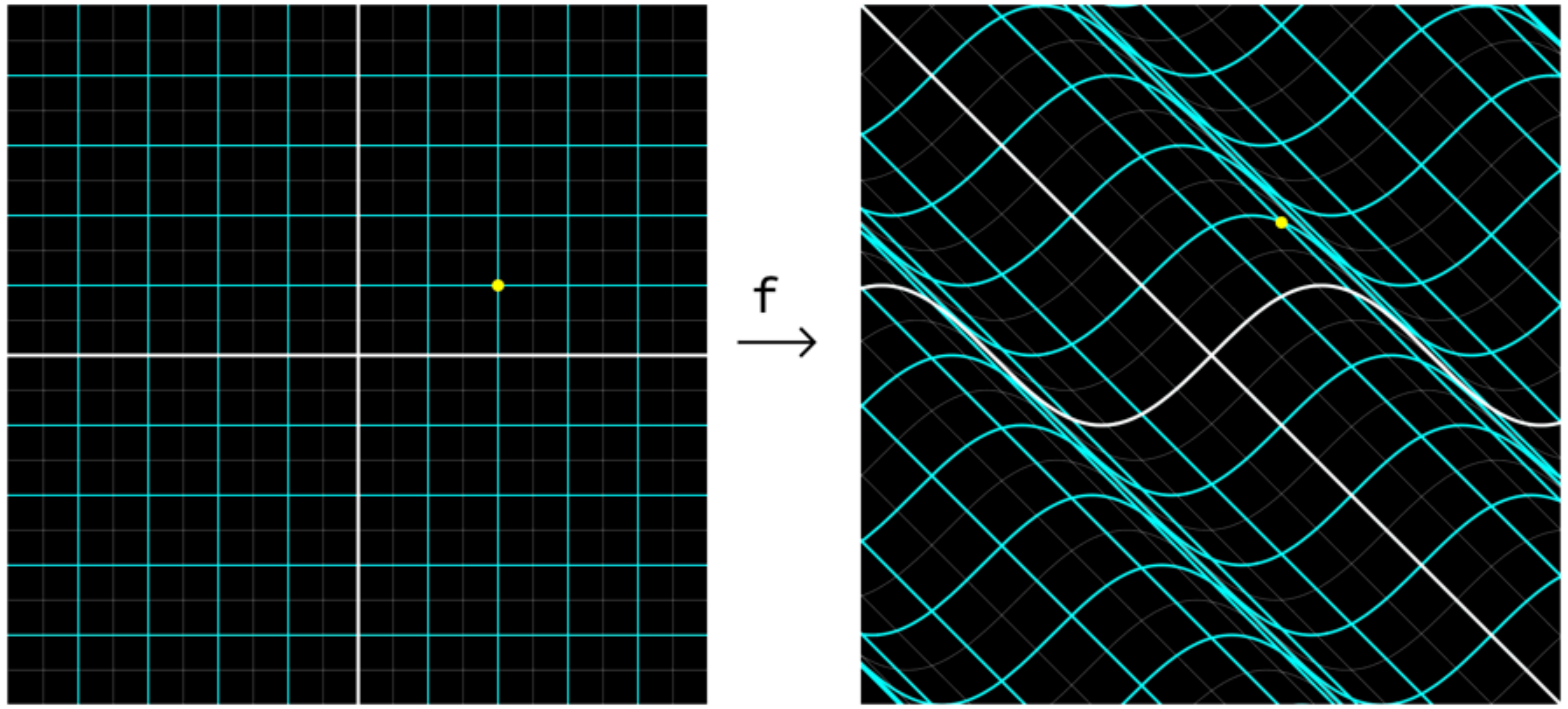
Often, the word **transformation** is used instead of function when talking about vectors.

$$f(x,y) = (x+y,\ 2y)$$



nice!

$$f(x,y) = (x-y,\ y+\text{Sin}[x])$$



too weird for this class